Performance indices for arbitrarily scaled rectangular cross-sections in bending stiffness design

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Abstract: Performance indices are presented for the selection of optimal rectangular beams in bending stiffness design. Previous studies have developed performance indices for only three design cases: proportional scaling of width and height, constrained height and constrained width. This paper extends the methodology of the performance index to any arbitrary direction of scaling. The performance index has the form E^q/ρ, where q is a function only of the scaling vector between two cross-sectional envelopes of different materials. The paper also presents a graphical method for determining the performance of rectangular beams in stiffness design. The performance index and the graphical method are applied to a design case study.

Keywords: performance index, arbitrary scaling, constrained design, optimal selection

NOTATION

A	cross-sectional area
b	width (m)
c_1	constant for stiffness depending on boundary conditions and load
D	cross-sectional envelope dimensions (b, h)
E	Young's modulus (GPa)
F	functional requirements
h	height (m)
I	second moment of area (m^4)
k	linear stiffness requirement (N/m)
L	length (m)
m	mass (Mg)
M	material parameters
p	performance index
q	power of the performance index
r_g	radius of gyration
S	shape of the cross-section
u	linear multiplicator of the widths
v	linear multiplicator of the heights
W	load (N)
z	constant for the second moment of area
δ	deflection (m)
ρ	material density (Mg/m^3)

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1 INTRODUCTION

1.1 Background

The minimization of mass for a given set of design requirements is often one of the most important goals in structural design. Low mass can lead to lower material costs, reduced environmental impact and improved technical performance such as better vibration characteristics. Reduction in mass is particularly critical in the aerospace and automotive industries.

One obvious way to achieve low mass in bending stiffness design is to select a material with a high ratio E/ρ. However, various authors [**1–3**] have shown that E/ρ does not always indicate which is the best material for a particular structure. The correct performance index for a particular application depends on the direction in which a section is scaled. In addition, the direction in which a section is scaled often depends on the geometric constraints on the design space (height constraint, width constraint, etc.).

For example, the ratio $E^{1/2}/\rho$ [**1–5**] indicates the best material when there is no space restriction and a section is scaled proportionally (see the Appendix). This ratio gives quite a different set of optimum materials compared to those provided by the ratio E/ρ. In addition, it has been shown that, if the cross-section is restrained in height, optimum materials are indicated by the ratio E/ρ [**2, 4, 6**], and if the cross-section is restrained in width, optimum materials are indicated by the ratio $E^{1/3}/\rho$ [**2, 4**]. These ratios illustrate that it is vitally important to understand how a cross-section is scaled in order to determine the best material to achieve low mass in bending stiffness design.

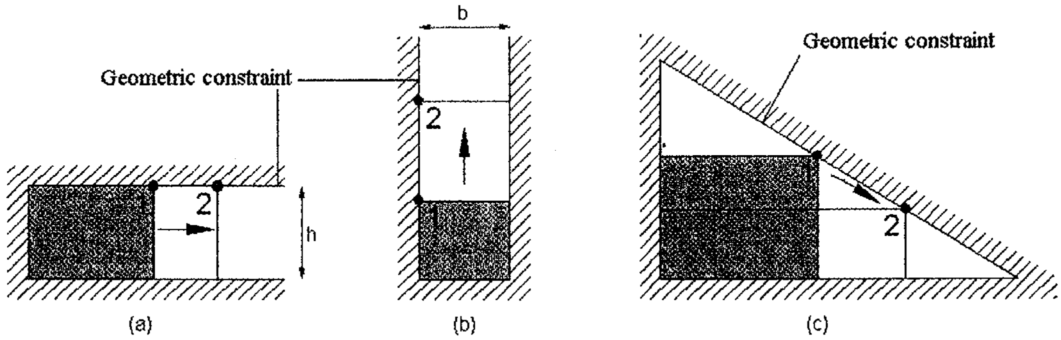

Fig. 1 Geometric constraints and the direction of scaling on the cross-sections of the beam: (a) height constraint; (b) width constraint; (c) slope constraint

The ratios E/ρ, $E^{1/2}/\rho$ and $E^{1/3}/\rho$ are all performance indices for low mass and given stiffness. However, these indices only represent three directions of scaling. In reality a section can be scaled in any direction.

This paper introduces the concept of the cross-sectional envelope dimensions, D, as a design variable in order to extend the method of performance indices to any arbitrary direction of scaling. It is shown that there is a general solution to the performance index E^q/ρ, where q is a function only of the scaling vector between two cross-sectional envelopes of different materials. A complete range of solutions for E^q/ρ is given. The paper presents a graphical method for structure selection, which gives the same results as the general solution of the performance index. Finally, a design example is carried out using both methods to show that geometrical constraints can have a significant effect on material selection.

1.2 Geometric constraints and direction of scaling

Three examples of different geometric constraints to the cross-section of a rectangular beam are shown in Fig. 1. These constraints are common in all branches of engineering. For example, in the design of a floor structure, there is often a height constraint as shown in Fig. 1a. In the design of a structure within a wall, there is often a width constraint as shown in Fig. 1b. In tightly constrained structures such as those found in the automotive and aerospace industries, it is not uncommon to have a constraint at an inclined angle as shown in Fig. 1c.

The influence of the geometrical constraints on the direction of scaling can be seen in Fig. 1. For example, where the height is constrained (Fig. 1a), it is only possible to change the width when considering different materials. Conversely, if the width is constrained (Fig. 1b), it is only possible to change the height of the beam. In the case of a geometric constraint at an inclined plane, as shown in Fig. 1c, both the height and width must change. There may be other reasons for scaling in certain directions, including availability of certain sizes and shapes or even a desire of the designer to keep a certain proportion to a section.

Arbitrary scaling of the height and width of the cross-section is shown in Fig. 2. Scaling of the cross-section can be specified by two linear multiplicators, u and v, where u is the relative change in width and v is the relative change in height of the cross-sectional envelope. In constrained height design, shown in Fig. 1a, $v = 1$ and corresponds to direction B in Fig. 2. In constrained width design, shown in Fig. 1b, and for direction C in Fig. 2, $u = 1$. When there is uniform proportional scaling of the cross-section, $u = v$. This is direction A in Fig. 2.

In the next section, the overall design methodology is described, together with a summary of different stiffness design conditions.

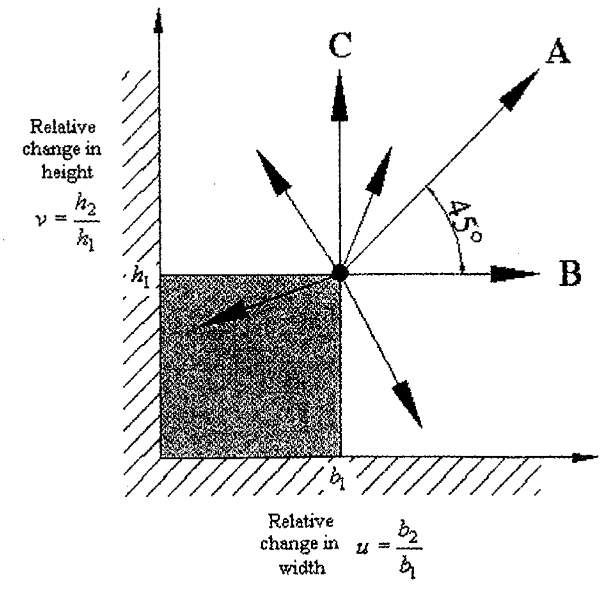

Fig. 2 Arbitrary direction of scaling of a beam cross-section: A, 45° scaling, $u = v$; B, horizontal scaling, $v = 1$; C, vertical scaling, $u = 1$

2 METHODOLOGY

2.1 Selection criteria

If the performance index, p, is measured by mass efficiency, then the performance of a structure is a function $f(\ )$ of at least four parameters:

$$p = f(F, D, S, M) \quad (1)$$

where F are the functional requirements, D describes the dimensions (width, b, and height, h) of the cross-sectional envelope, S is a description of the shape of the cross-section and M describes the material properties.

Generally, F is the design input, and D, S and M are generally the design variables. In the selection process, the best solution often involves a compromise between these three variables. In addition to the functional requirement F, the design variables must be compatible with the design constraints such as geometric constraints, material availability or shape availability. For example, the cross-sectional envelope D must be compatible with the geometric constraints as shown in Fig. 1.

Figure 3 illustrates a range of cases where different cross-sections meet the functional requirement F (the same stiffness). For example, Fig. 3a shows that, for a given material and cross-sectional shape (in this case the diamond family), the height and width of the cross-sectional envelope are changed while meeting the same stiffness, k, and consequently changes in the variable D occur. Figure 3b illustrates that, for a given cross-sectional shape, both the material, M, and the dimensions of the envelope, D, can be varied to meet the stiffness requirement. Figure 3c indicates that, for a given material, changes in the variable S usually cause a variation in the envelope. Figures 3d and e display other possible changes in the design variables.

2.2 Conditions for stiffness design

As Fig. 3 shows, it may be that in a particular design application there is a restriction on which design parameters D, S and M can be varied. Table 1 summarizes the different permutations of what design parameter can be varied. Whereas in each of the conditions the functional requirement remains the same, the other parameters D, S and M in equation (1) can be fixed or varied. Therefore, the performance index is a function of the free design variables. For example, in selection condition 1 the structures differ only

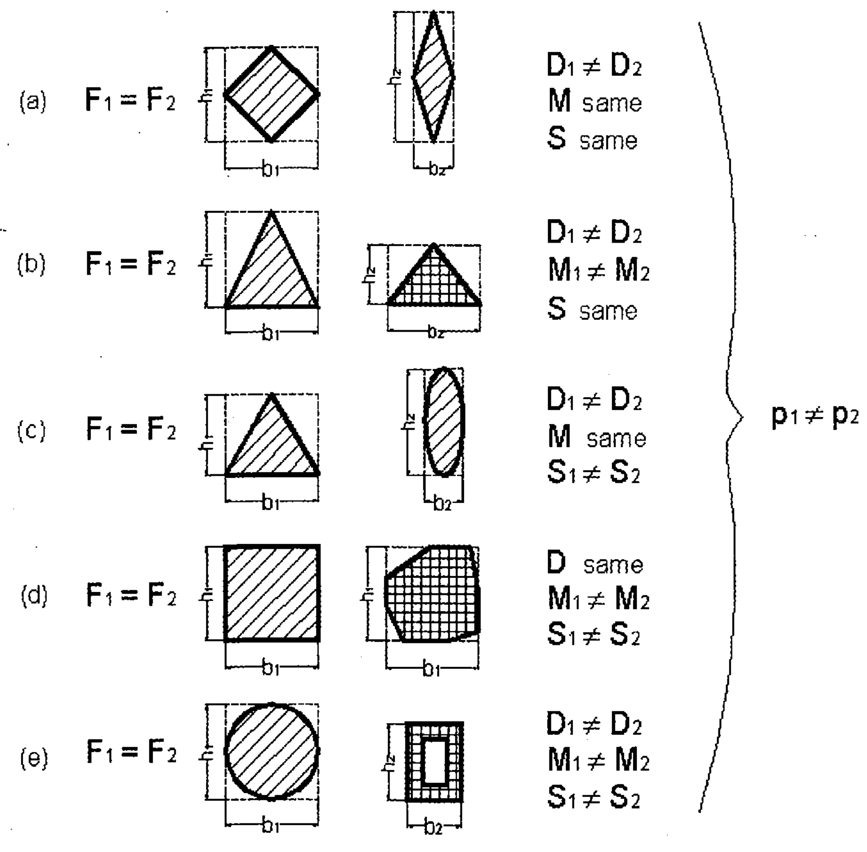

Fig. 3 Effect of variables on the performance of the structures meeting the same functional requirements (e.g. stiffness): (a) cross-sectional envelope variations; (b) variations in material and envelope; (c) changes in shape and envelope; (d) material and shape vary in the same envelope; (e) all the variables change

Table 1 Conditions for stiffness design

Selection condition	Functional requirement, F	Cross-sectional geometry: Cross-section envelope dimensions, $D(h, b)$	Cross-sectional geometry: Shape, S	Material, M	Performance index, $p = f(D, S, M)$
1 Cross-sectional envelope	Stiffness	Variable	Fixed	Fixed	$p = f(D)$
2 Cross-sectional envelope and material	Stiffness	Variable	Fixed	Variable	$p = f(D, M)$
3 Cross-sectional envelope and shape	Stiffness	Variable	Variable	Fixed	$p = f(D, S)$
4 Cross-sectional envelope and shape and material	Stiffness	Variable	Variable	Variable	$p = f(D, S, M)$

for the dimensions of the cross-sectional envelope S as one material M and one shape S are available. In condition 2 the selection occurs among structures of the same shape S but for different material M and cross-sectional envelope dimensions D. Conditions 3 and 4 consider further selection criteria.

This paper will deal with the first two selection conditions given in Table 1. In the next section a general expression of the performance index E^q/ρ will be derived for selection condition 2, i.e. when the dimensions of the cross-sectional envelope D (expressed by b and h) and the material M are variable. In Section 4, both selection conditions 1 and 2 will be examined using a graphical method.

The analysis used in this paper is based on the following assumptions:

1. Only bending stiffness design is examined.
2. Only homogeneous and isotropic materials are examined that obey Hooke's law and have a Young's modulus E that is the same in tension and compression.
3. The effects of local and general buckling are neglected.

3 RELATIVE PERFORMANCE INDEX E^q/ρ FOR AN ARBITRARY SCALED CROSS-SECTIONAL ENVELOPE OF DIFFERENT MATERIAL

3.1 Analysis

In bending stiffness design, the ratios E/ρ, $E^{1/2}/\rho$ and $E^{1/3}/\rho$ lead to the selection of the best material for constrained height, proportional scaling and constrained width. From this it can be seen that the direction of scaling affects the power to which E is raised.

As Table 1 shows, in selection condition 2 the performance index, $p = f(M, D)$ is a function of the free variables M and D. However, if the direction of scaling has been set in advance, such as in horizontal, proportional or vertical scaling, the variable D appears as the power of the Young's modulus E. Therefore, the aim of the following analysis is to find a general function $q = f(D)$ in order to derive a general expression of the performance index for arbitrary scaling.

For a given material, cross-section and set of design requirements, expressions for the mass, m, and the bending stiffness, k, of a structure are

$$\frac{m}{L} = \rho A \tag{2}$$

$$\frac{kL^3}{c_1} = EI \tag{3}$$

where c_1 is a constant depending on the boundary conditions, ρ is the density, A is the cross-sectional area, L is the length of all the structures and I is the second moment of area.

Consider the rectangle as the shape family of two structures 1 and 2 of different material. The width and the height of their cross-sectional envelopes D, which in this case coincide with the dimensions of the cross-sectional shape, are scaled by the two multipliers u and v, where

$$u = \frac{b_2}{b_1} \qquad v = \frac{h_2}{h_1} \tag{4}$$

The ratio of the masses m_1 and m_2 of two generic structures 1 and 2 of the same length, L, is

$$\frac{m_1}{m_2} = \frac{\rho_1 A_1}{\rho_2 A_2} \tag{5}$$

As maximizing the performance index minimizes the mass, combining equations (4) and (5), the ratio of the performance indices for structures 1 and 2 is

$$\frac{p_2}{p_1} = \frac{m_1}{m_2} = \frac{\rho_1}{\rho_2}\frac{1}{uv} \tag{6}$$

Expressions for u and v in terms of the design requirement are now sought. For bending stiffness design, both structures are required to meet the same stiffness requirement, where

$$E_1 I_1 = I_2 E_2 \tag{7}$$

and

$$\frac{E_1}{E_2} = \frac{I_2}{I_1} \tag{8}$$

The ratio of the second moments of area of the two structures can also be stated in terms of multipliers u and v, so that

$$\frac{I_2}{I_1} = uv^3 \tag{9}$$

When the height of the two structures is constrained, $v = 1$, and u is

$$u = \frac{I_2}{I_1} \tag{10}$$

Alternatively, when the width is constrained, $u = 1$, and v is given by

$$v = \left(\frac{I_2}{I_1}\right)^{1/3} \tag{11}$$

The performance indices for constrained height ($v = 1$) and width ($u = 1$) follow from equations (6), (8), (10) and (11), so that:

with $v = 1$

$$\frac{P_2}{P_1} = \frac{\rho_1}{\rho_2}\frac{I_1}{I_2} = \frac{\rho_1}{\rho_2}\frac{E_2}{E_1} \tag{12}$$

with $u = 1$

$$\frac{P_2}{P_1} = \frac{\rho_1}{\rho_2}\left(\frac{I_1}{I_2}\right)^{1/3} = \frac{\rho_1}{\rho_2}\left(\frac{E_2}{E_1}\right)^{1/3} \tag{13}$$

For arbitrary scaling conditions $u \neq 1$ and $v \neq 1$, a solution is sought such that

$$\frac{P_2}{P_1} = \frac{\rho_1}{\rho_2}\left(\frac{E_2}{E_1}\right)^{q} \tag{14}$$

where q is yet to be determined.

For these conditions

$$u = \left(\frac{I_2}{I_1}\right)^{\alpha}$$
$$v = \left(\frac{I_2}{I_1}\right)^{\beta} \tag{15}$$

and equations (1) and (9) are rewritten as

$$\frac{I_2}{I_1} = uv^3 = \left(\frac{I_2}{I_1}\right)^{\alpha + 3\beta} \tag{16a}$$

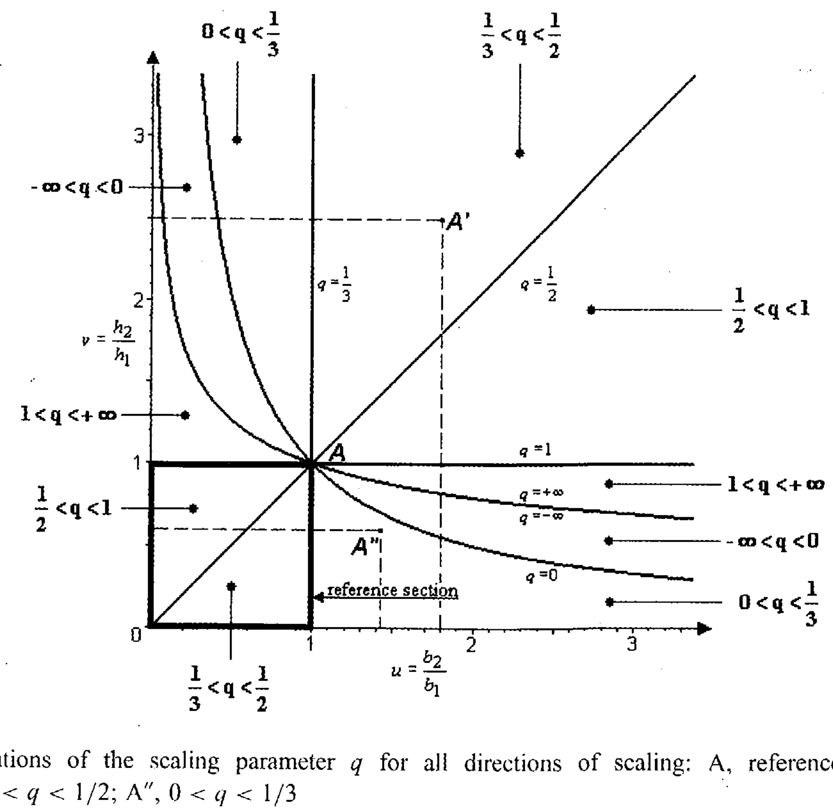

Fig. 4 Solutions of the scaling parameter q for all directions of scaling: A, reference section; A′, $1/3 < q < 1/2$; A″, $0 < q < 1/3$

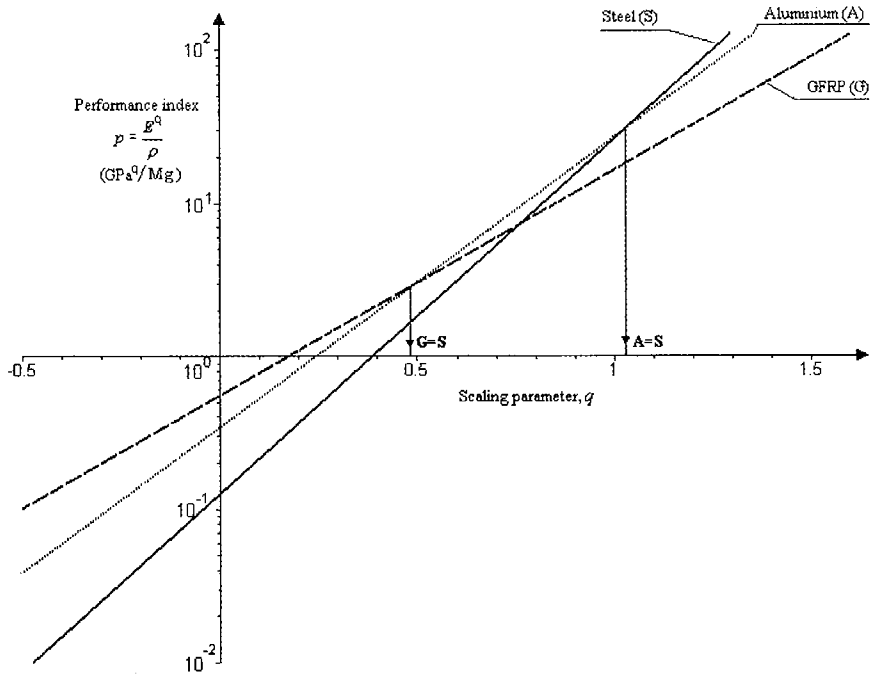

Fig. 5 Performance of the three materials for a range of values of q

and

$$\alpha + 3\beta = 1 \tag{16b}$$

From equation (15) the exponents α and β are

$$\alpha = \lg_{(I_2/I_1)} u = \lg_{(uv^3)} u$$
$$\beta = \lg_{(I_2/I_1)} v = \lg_{(uv^3)} v \tag{17}$$

The ratio of the performance indices p_2/p_1 follows from equation (6) using equations (15) to (17), so that

$$\frac{p_2}{p_1} = \frac{\rho_1}{\rho_2}\left(\frac{I_1}{I_2}\right)^q = \frac{\rho_1}{\rho_2}\left(\frac{E_2}{E_1}\right)^q \tag{18}$$

where

$$q = \alpha + 3\beta = \lg_{(uv^3)} uv = \frac{\ln uv}{\ln uv^3} \tag{19}$$

Equation (18) permits the performance indices for structures of similar cross-sectional shape with arbitrary cross-sectional envelopes to be compared. In particular, the exponent q represents a parameter that describes the scaling of the dimensions of the cross-sectional envelopes of structures using different materials. This is because each material requires a different space to meet the functional requirement k.

Figure 4 shows a plot of results for $q = f(u, v)$. These results are consistent with the previous values of $q = 1/3$, $q = 1/2$ and $q = 1$ for constrained width, proportional scaling and constrained height respectively. The figure shows that, for two curves $uv^3 = 1$ and $uv = 1$, q is unbounded (i.e. $q = \pm\infty$) and zero respectively. These results give an indication of the relative importance of Young's modulus and density. When q approaches zero, the density is more important in comparison with Young's modulus. In contrast, when q approaches infinity, Young's modulus is relatively important compared with the density. This immediately shows that the direction of scaling has a very important effect on material selection.

Furthermore, Fig. 4 shows examples of arbitrary scaled rectangular sections of different materials. If a reference structure 1, for instance, has a cross-section of unit dimensions, and, according to the stiffness requirement, is rescaled so that $u = 1.8$ and $v = 2.5$, then point A moves to point A$'$ and $1/2 < q < 1$. Alternatively, if point A moves to point A$''$, then also $0 < q < 1/3$. As well as regions defined by $1/2 < q < 1$ and $0 < q < 1/3$, distinct regions for other ranges of q are shown.

In the next section it is shown that materials with high Young's modulus, such as steel, perform relatively better for high values of q. Furthermore, useful q ranges are presented where one material provides lower mass compared with others.

Table 2 Properties of materials

Material	Young's modulus, E (GPa)	Density, ρ (Mg/m^3)
Steel	210	7.9
Aluminium	79	2.9
GFRP	30	1.8

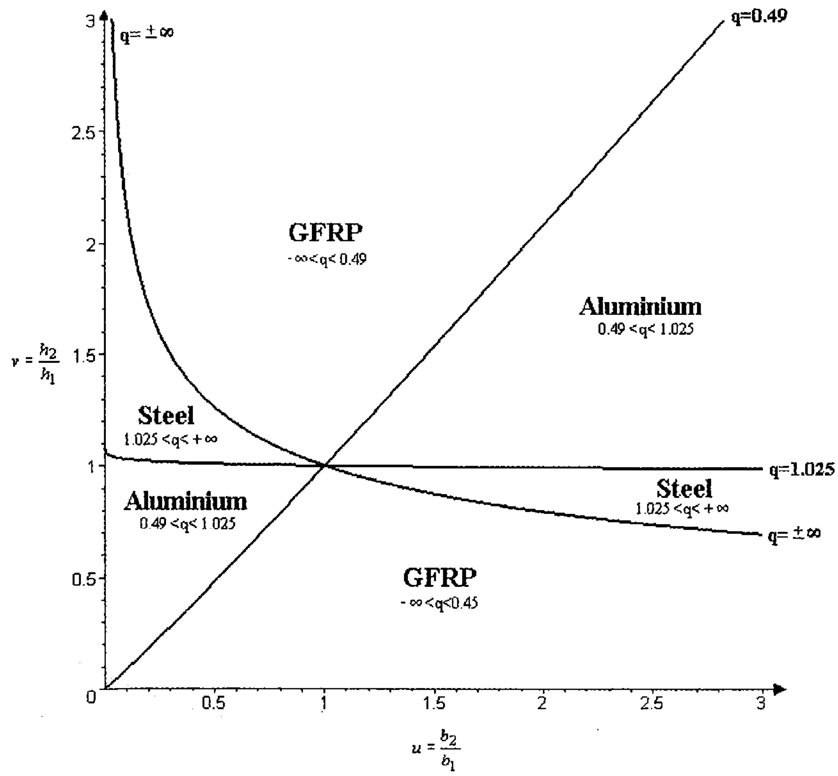

Fig. 6 Limiting material regimes for steel, aluminium and GFRP

3.3 Limiting material regimes

The general solution to the performance index [equations (18) and (19)] enables a comparison to be made between the performance of different materials for any direction of scaling. In stiffness design, the performance index p for arbitrary scaling with the aim of minimizing mass is expressed by

$$p = \frac{E^q}{\rho} \qquad (20)$$

where q is a function of the multipliers u and v.

Examples of a full range of solutions for this performance index for three materials [aluminium, steel and glass fibre reinforced plastic (GFRP)] are shown in Fig. 5. The performance index has been plotted as a function of the scaling parameter q using values of E and ρ given in Table 2.

When the direction of scaling is known in a design task, q can be calculated from equation (19) and the relative performance of different materials can be immediately determined from Fig. 5.

The intersection points of two curves in Fig. 5 represent values of the scaling parameter q where both materials perform equally. Thus, when q, for example, is greater than 1.025, steel cross-sections have a better performance index than aluminium and also GFRP cross-sections. When the scaling parameter q is less than 0.49, all GFRP rectangular cross-sections provide the best performance compared with aluminium and steel.

The parameter q is the scaling parameter. Variations in v from variation in u, for a given value of q, can be found by inverting equation (19) so that

$$v = u^{(1-q)/(3q-1)} \qquad (21)$$

Curves of special q values for which two materials have the same performance index can be plotted and then limiting material regions mapped. These special values of q, obtained from Fig. 5, are plotted in Fig. 6 using equation (21).

Figure 6 shows regions where the performance of one material is relatively better compared with the others. For example, in bending stiffness design all the rectangular cross-sections manufactured from aluminium provide the best performance index in the region where $0.49 < q < 1.025$. Alternatively, all cross-sections scaled so that they lie in the GFRP region provide minimum mass compared with steel and aluminium. A later section shows an example of design application where the limiting material regimes shown in Fig. 6 are used without the need of any calculation.

4 GRAPHICAL METHOD FOR SELECTING OPTIMAL STRUCTURES IN BENDING STIFFNESS DESIGN

The performance index in the form of E^q/ρ does not allow the selection of structures of the same material. This is because the parameters u and v in the expression for q are multipliers of the dimensions of different material cross-sectional envelopes. However, in bending stiffness design, a simple selection can occur among structures of the same material and same shape. This is the case of condition 1 of Table 1. Consequently, a graphical procedure is presented in the following sections for the selection condition 1, and this is extended to condition 2 of Table 1.

4.1 Selection condition 1: the same cross-sectional shape and material

For a given material, cross-section and set of design requirements, expressions for the mass, m, and the bending stiffness, k, of a structure are expressed by equations (2) and (3). By replacing the area, A, in equation (2) and the second moment of area, I, in equation (3) as functions of the width, b, and depth, h, of the cross-sectional envelope, the mass and the stiffness of a rectangle are

$$\frac{m}{L} = bh\rho \tag{22}$$

$$\frac{12kL^3}{c_1} = bh^3E \tag{23}$$

Figure 7 illustrates curves obtained using equations (22) and (23), where the only variables are the height and width of the structure. Curves A and B represent all rectangular cross-sections of equal mass and equal stiffness respectively. Each curve corresponds to the mass and the stiffness of rectangle 1. Rectangle 2 has the same stiffness and has lower mass than rectangle 1. For a stiffness requirement k, the only possible scaling of the cross-section is along curve B. Consequently, simple horizontal scaling $v = 1$, vertical scaling $u = 1$ and proportional scaling $u/v = \text{constant}$ are not feasible.

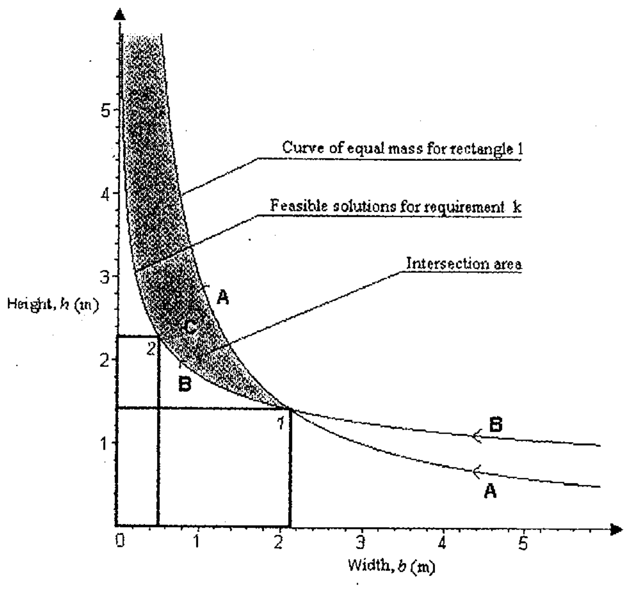

Fig. 7 Curves for equal stiffness (B) and mass (A)

Compared with envelope 1, all the rectangles under curve A have lower mass. Envelopes above curve B are stiffer. The shaded area C in Fig. 7 represents solutions for all possible rectangles that are both stiffer and lighter than rectangle 1.

In bending stiffness design, the performance index, p, of a structure can be seen as the ratio of equations (3) and (2):

$$p = \frac{k}{m} = \frac{c_1}{L^4}\frac{E\,I}{\rho\,A} \tag{24}$$

Then

$$p = \frac{1}{m} = \underbrace{\left(\frac{c_1}{kL^4}\right)}_{F}\underbrace{\left(\frac{E}{\rho}\right)}_{M}\underbrace{(r_g^2)}_{S,D} \tag{25}$$

where r_g is the radius of gyration of the section.

Equation (25) is divided into three groups: F collects the common parameters of the structure and the stiffness requirement, M the material properties and D with S the cross-sectional envelope dimensions and the shape. However, in stiffness design (unless some assumptions are made about the dimensions of the cross-sectional shape), the second moment of area, I, is not separable from the material property. Young's modulus determines the space that cross-sections must occupy in order to meet the functional requirement k.

For a given stiffness k, the performance of a rectangle is assessed by equation (24) together with the stiffness requirement equation (23), so that

$$p = \left(\frac{E}{\rho}\right)\left(\frac{h^2}{12}\right) \tag{26}$$

where

$$h = \left(\frac{1}{bE}\right)^{1/3} \tag{27}$$

The performance index p [equation (26)] is a function of the height of the rectangle. The height is also a function of both b and E [equation (27)] (Only if there is a width constraint can equation (27) be substituted into equation (26), and the performance index take the known form $E^{1/3}/\rho$.)

Figure 8 shows graphical solutions for equations (26) and (27). The performance indices for rectangular sections 1 and 2 are p_1 and p_2. It is evident that, since p_2 is greater than p_1, rectangle 2 provides lower mass than rectangle 1. An important feature of Fig. 8 is illustrated by referring to the design space given by the triangle OXY. The intersection of the line XY with curve B indicates that there are two possible solutions, rectangles 1 and 2, that satisfy the stiffness requirement. However, rectangle 2 provides a lower mass than rectangle 1.

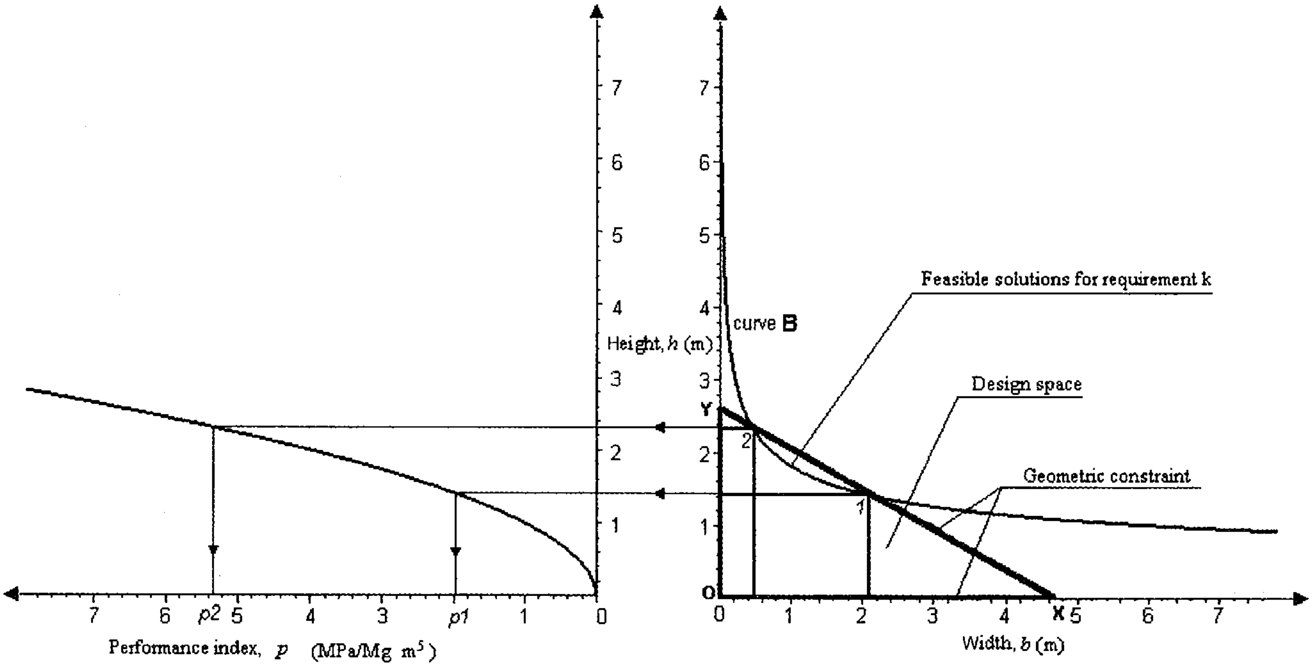

Fig. 8 Combined performance graph for rectangular sections of the same material

4.2 Selection condition 2: the same cross-sectional shape but different materials

When both D and M are variables, curve B in Figs 7 and 8 differs for each material. This is illustrated in Fig. 9, where the functional requirement, i.e. stiffness, is the same for aluminium and steel. Materials with high E, for example steel, provide a benefit in terms of space. The performance of materials such as aluminium and steel can be assessed through the combined performance graph.

In this selection condition of stiffness design, the direction of scaling can occur from all the points of each stiffness curve; for example, horizontal (from point B to B′), vertical (from C to C′) and proportional scaling (from A to A′) are feasible.

It is evident that, in the case of the rectangles E and F in Fig. 9, the performance index p_A is greater than p_S. Therefore, the aluminium rectangular cross-section provides lower mass than steel, even though the steel occupies less space. Different cross-sections from E and F can provide opposite results. This graphical approach is equivalent to the analysis presented in Section 3.

The performance of materials depends on how the cross-sectional envelopes are scaled. The next section shows a

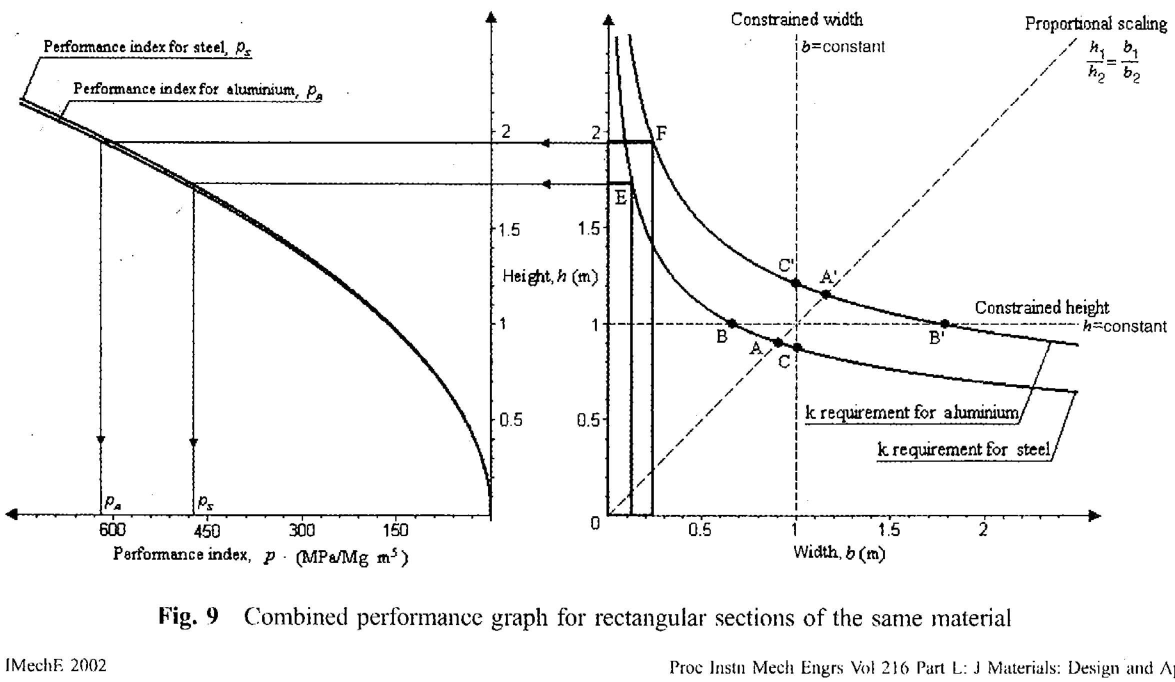

Fig. 9 Combined performance graph for rectangular sections of the same material

design application where geometrical constraints impose different directions of scaling, which in turn affect the choice of material.

5 DESIGN CASE STUDY

A design case study has been carried out using both selection methods: the limiting material regimes chart and the combined performance graph. The final results are the same and are confirmed by the numerical calculations presented in Tables 4 and 5.

The example is a 4 m cantilever that must support an end load of 1000 N with an allowable end deflection, δ, of 3.2 cm. This is illustrated in Fig. 10.

Steel and aluminium are the materials available in rectangular shapes, and design data are summarized in Table 3. Two design constraints are examined: a height constraint is imposed, and then a sloped constraint with $q = 1.44$. These conditions are shown in Fig. 10.

5.1 Limiting material regimes graph

Figure 5 illustrated that, for steel and aluminium, when $q > 1.025$, steel performed better than aluminium. The limiting regimes for the two materials are again illustrated in Figs 11 and 12. In the first case, there is height constraint, $q = 1$ and a steel rectangular cross-section, providing the same stiffness, lies within the region where

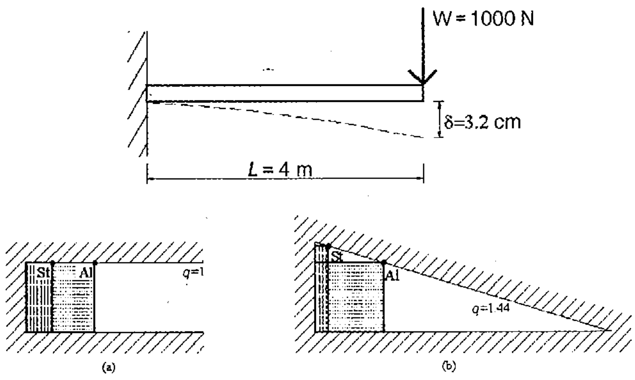

Fig. 10 Cantilever and its cross-sections in two different constrained conditions: (a) height constraint ($q = 1$); (b) sloped constraint ($q = 1.44$)

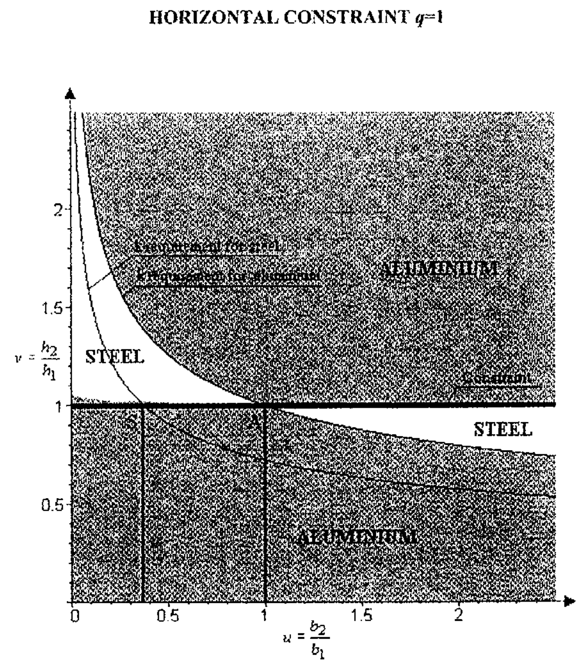

Fig. 11 Limiting material regimes graph for a horizontal constraint ($q = 1$)

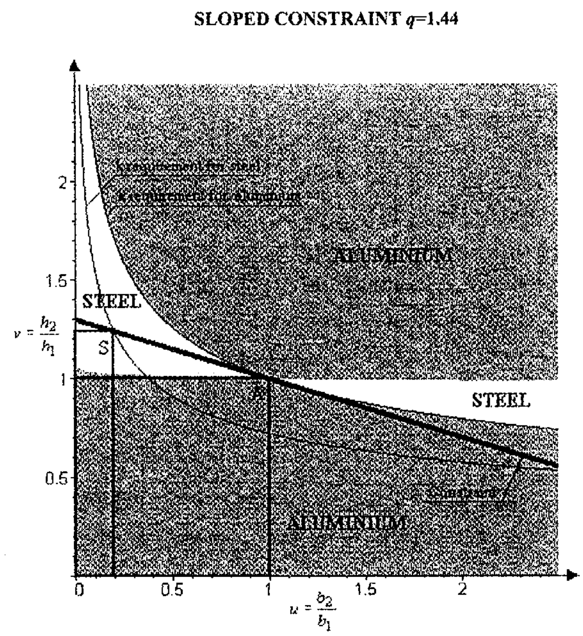

Fig. 12 Limiting material regimes graph for a sloped constraint ($q = 1.44$)

Table 3 Design data for case study

	Young's modulus, E (GPa)	Density, ρ (mg/m^3)	Load, W (N)	Deflection, δ (m)	Stiffness requirement, k (N/m)	Length, L (m)	Boundary condition constant, c_1
Aluminium	79	2.9	1000	0.0324	30859	4	3
Steel	210	7.9	1000	0.0324	30859	4	3

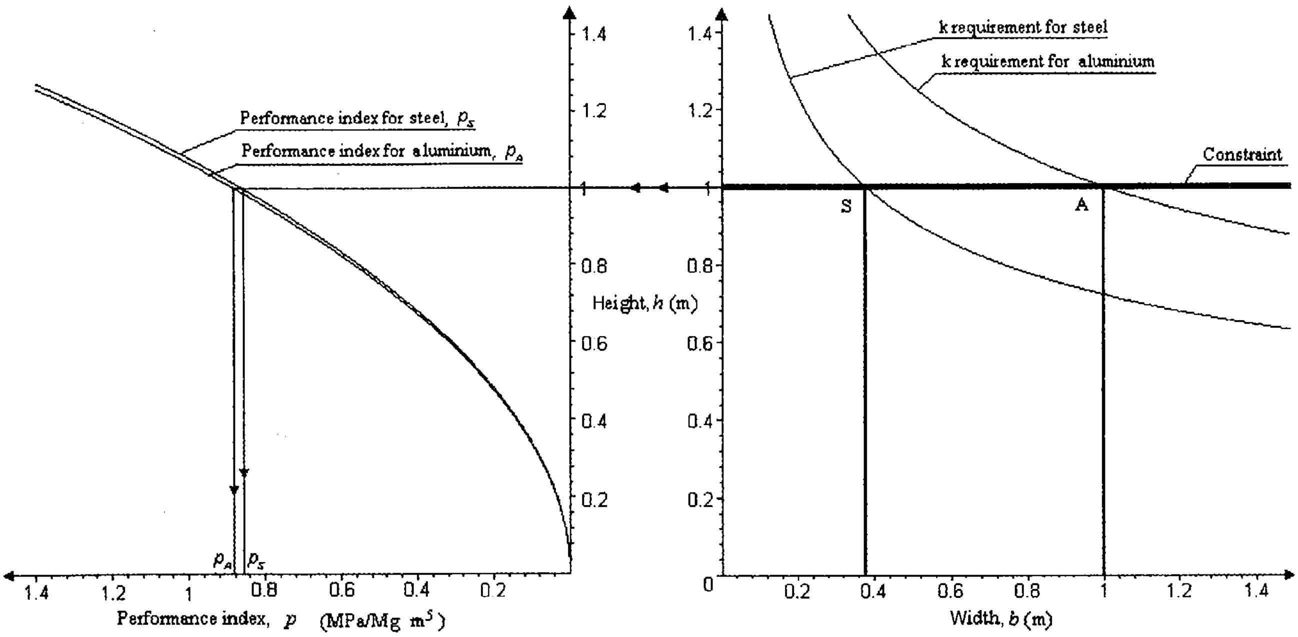

Fig. 13 Combined performance graph for a horizontal constraint

aluminium is better. Therefore, aluminium provides lower mass.

In the second case there is a sloped constraint, (Fig. 12). The same aluminium section 1 × 1 m is compared with a steel rectangular cross-section. The sloped constraint dictates the direction of scaling. Figure 12 illustrates that this constraint intersects the region where steel performs better than aluminium. This is in contrast to the first case, and the steel cross-section has lower mass than aluminium.

5.2 Combined performance graph

The combined performance graphs for the two constraints $q = 1$ and 1.44 are illustrated in Figs 13 and 14 respectively. For horizontal constraint the graphical solution shown in Fig. 13 demonstrates that that the performance of the aluminium cross-section is marginally better than that of steel. The numerical results in Table 4 show that the mass saving using aluminium is just 2 per cent. Prescribing a

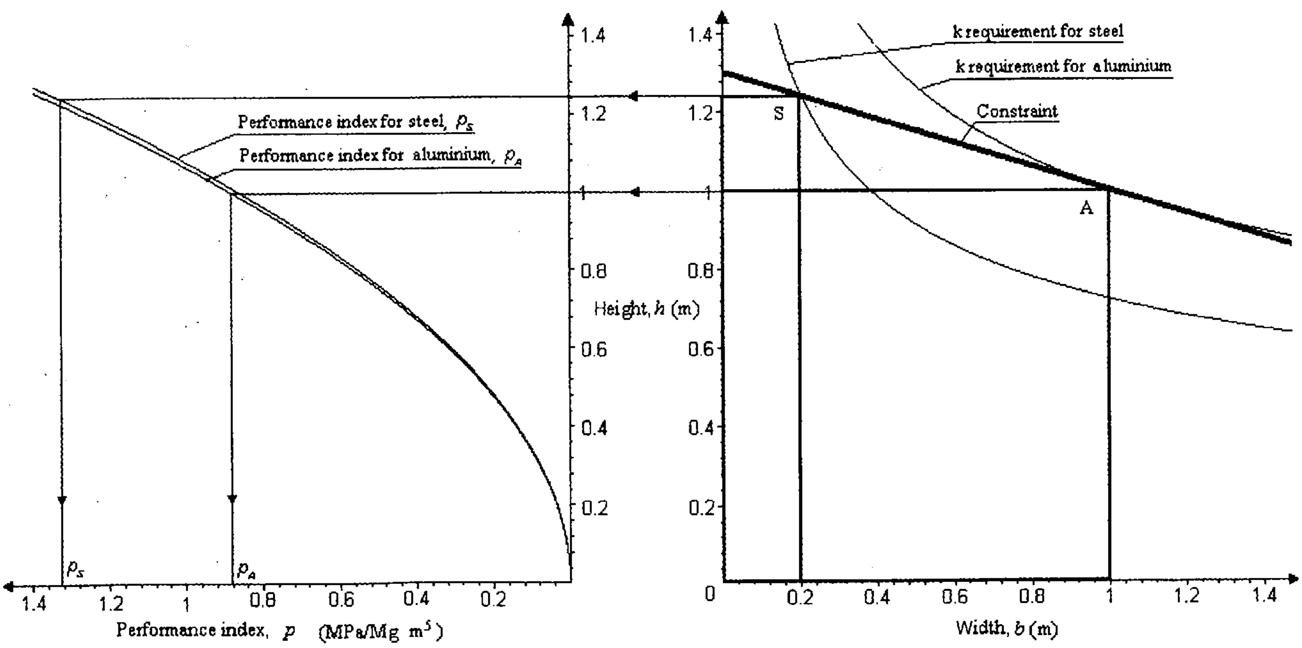

Fig. 14 Combined performance graph for a sloped constraint

Table 4 Numerical results for height constraint ($q = 1$)

	Width, b (m)	Height, h (m)	Height multiplicator, u	Width multiplicator, v	Power of performance index, q	Stiffness, k (N/m)	Performance index, $p = E^q/\rho$	Mass, m (Mg)	Mass saving (%)
Aluminium	1.00	1.00			1.00	30859	27.2	11.60	2
Steel	0.376	1.00	0.376	1.00	1.00	30859	26.5	11.89	

Table 5 Numerical results for sloped constraint ($q = 1.44$)

	Width, b (m)	Height, h (m)	Width multiplicator, u	Height multiplicator, v	Power of performance index, q	Stiffness, k (N/m)	Performance index, $p = E^q/\rho$	Mass, m (Mg)	Mass saving (%)
Aluminium	1.00	1.00			1.44	30859	186.3	11.60	
Steel	0.197	1.24	0.197	1.24	1.44	30859	279.6	7.731	33

sloped constraint leads the choice of steel to provide minimum mass. Table 5 shows that a remarkable mass saving of 33 per cent is achieved.

This result demonstrates that geometric constraints can have a very important influence on what is an optimal material. Using the performance indices E/ρ, $E^{1/2}/\rho$ and $E^{1/3}/\rho$ always indicates that aluminium is better than steel. However, the design example shows that, when there is a sloping height constraint, steel can be better than aluminium. This case study demonstrates also the importance of having a general solution of the performance index.

6 CONCLUDING REMARKS

Earlier work examined material performance indices for proportional scaling, and for height and width constraint alone. Geometric constraints on the design space will restrict the direction in which a section can be scaled. It is therefore important to consider arbitrary scaling of the dimensions of the envelope of the cross-section. This paper began by introducing the dimensions of the envelope, D, of the cross-section and its shape, S, as design variables. Different permutations of the design parameters were considered. This paper had focused on varying the envelope, D, and material, M.

Two alternative methods for selecting light structures in bending stiffness design have been presented. The first procedure extends the approach of the performance index in the form of ratio E^q/ρ for arbitrary scaled cross-sections of the same shape but different material.

The second method is a graphical solution that considers arbitrary scaling of the same material and arbitrary scaling of different materials.

The first method is used to provide a diagram illustrating limiting material regions where one material performs better than other materials.

Both methods were then used to determine the best performance (i.e. minimum mass) of a cantilever subjected to an end load. Contrary to conventional material performance indices, it has been demonstrated that steel can perform better than aluminium in providing minimum mass for the same bending stiffness.

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APPENDIX

Derivation of the performance index $E^{1/2}/\rho$ in bending stiffness design

In bending stiffness design, a beam of the type shown in Fig. 10 must meet a stiffness requirement of the form

$$k = \frac{W}{\delta} \tag{28}$$

From elasticity theory, the stiffness of the beam is given by

$$k = c_1 \frac{EI}{l^3} \tag{29}$$

where c_1 is a constant that depends on the details of the load and boundary conditions.

The mass of the beam is given by

$$m = \rho A l \qquad (30)$$

To derive a performance index it is necessary to replace the free variable A with the functional requirements. The area A in equation (30) can be replaced by rearranging equation (29) as a function of A, assuming that

$$I = \int_A y^2 \, \mathrm{d}A \qquad (31)$$

For a solid cross-sectional area with two symmetric axes, the second moment of area can be written as

$$I = zbh^3 = zh^2 A \qquad (32)$$

where z is a constant of the cross-sectional shape.

If the cross-sectional area is assumed to be an equiaxed shape (circle, square, etc.), the second moment of area can be expressed as a function only of the area. Therefore, the term h^2 in equation (32) can be replaced by A, and then I takes the form

$$I = zA^2 \qquad (33)$$

Combining the previous equation (33) with equation (29) after rearrangement, the mass of a beam that meets the stiffness requirement, k, can be expressed as

$$m = \left(\frac{c_1 k l^5}{z}\right)^{1/2} \left(\frac{\rho}{E^{1/2}}\right) \qquad (34)$$

The selection of the best material occurs by the evaluation of the following performance index:

$$p = \frac{E^{0.5}}{\rho} \qquad (35)$$

Since equation (33) is based on the assumption that there is proportional scaling of the section, the performance index $E^{1/2}/\rho$ is only valid for sections that are proportionally scaled.