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A general solution to the material performance index for bending strength design

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Abstract

This paper presents a general solution to the material performance index for the bending strength design of beams. In general, the performance index for strength design is ρ_f^q/ρ where σ_f is the material strength, ρ is the material density and q is a function of the direction of scaling. Previous studies have only solved q for three particular cases: proportional scaling of width and height ($q = 2/3$), constrained height ($q = 1$) and constrained width ($q = 1/2$). This paper presents a general solution to the exponent q for any arbitrary direction of scaling. The index is used to produce performance maps that rank relative material performance for particular design cases. The performance index and the performance maps are applied to a design case study.

Keywords: Performance index; Direction of scaling; Constrained design; Strength design

1. Introduction

The minimisation of mass is a common objective in the design of structures. Low mass can help produce low cost, high technical performance and reduced environmental impact. One way of minimising mass is to define a material index and select a material with the most optimum value of the index. Shanley [1], Cox [2], Parkhouse [3] and Charles and Crane [4] were some of the first authors to define material indices for minimising mass. Ashby and Weaver [5–7] have more recently refined the methodology. In general, the performance index for strength design is given by

$$p = \sigma_f^q/\rho, \quad (1)$$

where σ_f is the material strength, ρ is the material density and q is a function of the direction of scaling.

For a particular structural problem, the lightest material is identified by maximising the value of the performance index, p. Previous studies on strength design have produced solutions for q for three specific cases: proportional scaling of width and height ($\sigma_f^{2/3}/\rho$) constrained height (σ_f/ρ) and constrained width ($\sigma_f^{1/2}/\rho$) [8]. However, a cross-section can be scaled in an infinite number of directions and there is a need for designers to be able to calculate the material index for any direction of scaling.

This paper presents a general solution to the exponent of the performance index q for any arbitrary direction of scaling. The paper considers the case of bending strength of beams. When the direction of scaling is known in a particular design problem, the value of q and hence performance index can be calculated. The performance index is used to produce maps that rank the relative performance of different materials. The design maps help the designers to visualise the

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Nomenclature

A	cross-sectional area	p	performance index
B	width (m)	q	power of the performance index
D	cross-section envelope dimensions (B H)	S	shape of the cross-section
E	Young's modulus (GPa)	u	linear multiplier of the widths
F	functional requirements	v	linear multiplier of the heights
H	height (m)	P	load (N)
I	second moment of area (m^4)	Z	section modulus (m^3)
L	length (m)	y_m	furthest distance of a cross-section fibre from neutral axis
m	mass (Mg)		
M	material properties	ρ	material density (Mg/m^3)
M_f	moment failure requirement	σ_f	material strength in bending (MPa)

effect of the direction of scaling in material selection. In the last section of the paper, the performance index and the performance maps are applied to a design case study.

2. Scaling of cross-sections

2.1. Different directions of scaling

Fig. 1 shows different directions of scaling of a generic cross-section. When the height to depth ratio is kept constant, the cross-section is proportionally scaled along direction Z from point A to point B. A horizontal direction of scaling in direction X means that the height remains constant. If there is vertical scaling in direction Y, then the width remains constant. The directions X, Y, Z are just three directions of scaling. However, there are an infinite number of directions for scaling a cross-section. For example, the directions I, J and K shown in Fig. 1 are three arbitrary directions of scaling.

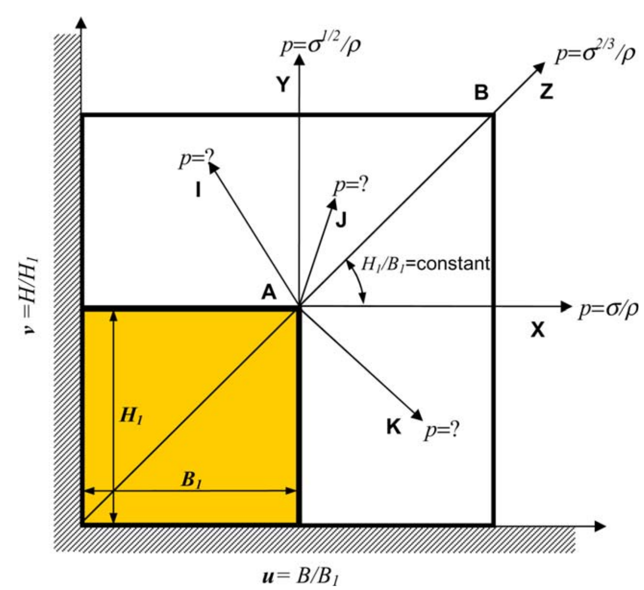

Fig. 1. Scaling of a beam cross-section in different directions. Directions X, Y, Z are horizontal, vertical and proportional scaling respectively. I, J, K are examples of scaling in arbitrary directions.

In strength design, the performance indices that indicate optimal material for horizontal, proportional and vertical scaling are known. However, the expression of p is unknown when there is an arbitrary direction of scaling, such as directions I, J, K as shown in Fig. 1. The main objective of this paper is to find a general solution σ_f^q/ρ to select the best material for a cross-section scaled in any direction such as I, J, K.

The direction of scaling of a cross-section can be specified by two linear multipliers, u and v, where u is the change in width and v is the change in height of the cross-section. For example, $v = 1$ describes a horizontal scaling and $u = 1$ describes a vertical scaling. Proportional scaling of the cross-section (direction Z) is described by $u = v$.

2.2. Reasons for scaling

Structural design generally involves geometrical constraints on the design space and this restricts the directions in which a section can be scaled. Geometrical constraints exist because the components in modern systems and equipment are tightly integrated and a limited allocation of space is given to structural members [9,10]. Fig. 2 shows three common examples of geometric constraints on the cross-section of a rectangular beam. In the design of floor structures, there is often a height constraint which imposes a horizontal direction of scaling as shown in Fig. 2(a). In the design of a wall structure, a width constraint forces the cross-section to be vertically scaled as shown in Fig. 2(b). In tightly constrained elec-

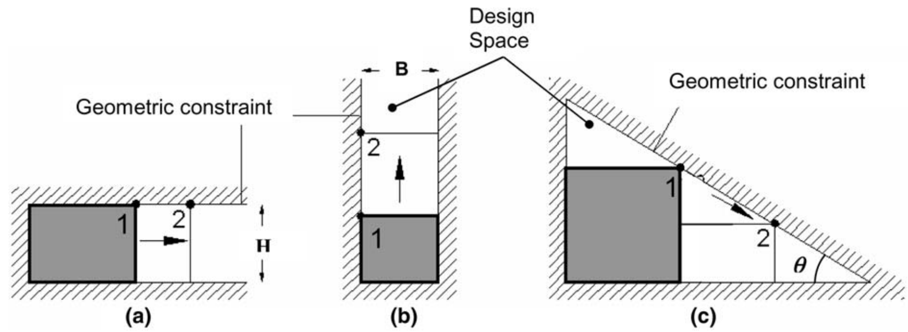

Fig. 2. Geometric constraints on the cross-sections of a beam: (a) height constraint, (b) width constraint, (c) negative slope constraint.

tromechanical structures it is not uncommon to have a geometrical constraint which results in a scaling direction at an inclined angle as shown in Fig. 2(c).

In addition to geometrical constraints, there may be other reasons for scaling in certain directions, such as availability of certain shapes or because of a certain type of loading. For example, proportional scaling is appropriate when bending loads are applied to a cross-section in more than one principal direction. In this case, a uniform increase (proportional scaling) of the section size is the best type of scaling for minimising mass.

3. Methodology

In order to derive a general solution to the material performance index, σ_f^q/ρ, a new approach for modelling the mass-efficiency of a structure is adopted as defined by Pasini [11,12]. In the next section, we first introduce the variables used to model the geometry of a cross-section. Then we illustrate how these variables influence the solution of the performance index.

3.1. Variables for modelling geometry

The methodology introduces two variables to describe the geometry of a cross-section: D and S. The variable, D, models the rectangular envelope of a section and specifies the sizes of that cross-section. The variable S models the shape of a cross-section within the envelope, D. Analytical definitions of S have been defined by Pasini [11]. One advantage of the method is that the separate contributions of material and shape can be systematically identified.

In addition to the geometric variables (D and S), the designer also has to select the material attribute variables, M. Therefore, a designer maximises the performance by selecting the variables: M, S, and D subject to the constraints of the problem. In Fig. 3, M, S and D are shown for two cross-sections. This example shows that S and M change while D remains constant because the cross-sections are not scaled.

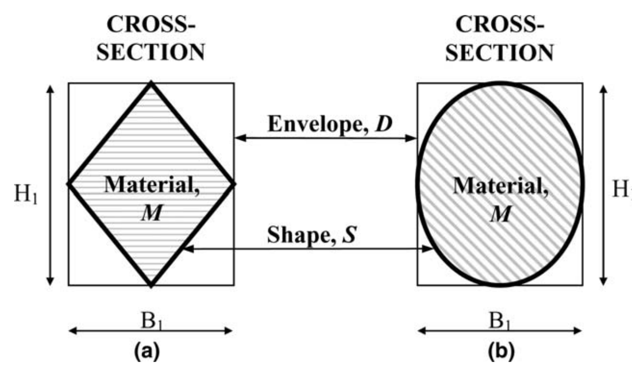

Fig. 3. The variables M, S and D describe the material, shape and space variables of a cross-section.

3.2. The performance index

In general, the main task of the designer is to choose the best combination of the variables M, S and D in order to maximise performance for a given set of requirements, F. The variables must be compatible with the geometric constraints, such as those applied to the envelope, D. If the performance index p is a measure of mass efficiency, then the performance of a structure is a function $f(\ )$ of at least four groups of parameters

$$p = f(F, D, S, M). \quad (2)$$

This paper examines the selection of the optimum material for structures which meet a given failure moment requirement. The general expression of the performance index σ_f^q/ρ is presented for scaled cross-sections with the same shape. We assume that the structures are beams subjected to bending and that the materials are homogenous, isotropic and obey Hooke's law. The bending failure moment is assumed to be the moment that causes the onset of yield. In the case of composites, the yield point may be the same as the ultimate tensile strength and so when comparing results in this paper, account should be taken that the failure mode for composites may be more severe. The focus of this paper is the selection of materials to minimise mass. However, the principles illustrated in this paper could be applied to case studies where the performance criterion is that of minimising cost.

4. Solution of Eq. (1)

The aim of the following analysis is to find a general function $q = f(D)$ for σ_f^q/ρ which can be applied for any scaling direction.

4.1. Scaling of the section modulus

In order to produce a general solution for the performance index it is necessary to define how the section modulus scales as a function of u and v. For a given material and set of design requirements, the mass m and the bending moment M_f which causes failure in a cross-section are given by

$$m = \rho AL, \tag{3}$$

$$M_f = \sigma_f Z, \tag{4}$$

where ρ is the density, A is the cross-sectional area, L is the length of all the structures, σ_f is the material strength and Z is the section modulus, i.e. $Z = I/y_m$ with I second moment of area and y_m the furthest distance of the outer fibre from the neutral axis.

Consider two beams, 1 and 2, with different materials, $M_1 \neq M_2$, but the same cross-section shape, $S_1 = S_2$ and the same length $L_1 = L_2$. In this analysis, the shape of the cross-section is assumed to be a rectangle and therefore the shape completely fills the envelope. As described in Section 2.1, the direction of scaling between two cross-sections is expressed by the multipliers u and v. The values of u and v are given by

$$\begin{cases} u = \frac{B_2}{B_1}, \\ v = \frac{H_2}{H_1}, \end{cases} \tag{5}$$

where B and H are the width and the height of a cross-section.

The ratio of the masses m_1 and m_2 of the beams, 1 and 2, of the same length L, is

$$\frac{m_1}{m_2} = \frac{\rho_1}{\rho_2}\frac{A_1}{A_2}. \tag{6}$$

Since mass is minimised by maximising the performance index (Eq. (1)), combining Eqs. (5) and (6) gives the ratio of the performance indices for beams 1 and 2

$$\frac{p_2}{p_1} = \frac{m_1}{m_2} = \frac{\rho_1}{\rho_2}\frac{1}{uv} \tag{7}$$

In order to identify individual performance indices, p_1 and p_2, it is necessary to eliminate u and v from Eq. (7). Therefore, we now seek expressions for u and v in terms of the design requirement. In strength design, both structures are required to meet the same moment failure requirement, M_f, and therefore

$$\sigma_{f1}Z_1 = \sigma_{f2}Z_2 \tag{8}$$

and

$$\frac{Z_1}{Z_2} = \frac{\sigma_{f2}}{\sigma_{f1}}. \tag{9}$$

The multipliers u and v can also be used to define the ratio of the section modulus as follows:

$$\frac{Z_2}{Z_1} = uv^2 \tag{10}$$

Eq. (10) is now used to solve Eq. (7) for different directions of scaling.

4.2. Horizontal scaling: $v = 1$

When the height of the two structures is constrained, $v = 1$, and from Eq. (10) the variable u is given by

$$u = \frac{Z_2}{Z_1}. \tag{11}$$

The ratio of the performance indices, Eq. (7), is given by combining expressions (9) and (11)

$$\frac{p_2}{p_1} = \frac{\rho_1}{\rho_2}\frac{Z_1}{Z_2} = \left(\frac{\rho_1}{\sigma_{f1}}\right)\left(\frac{\sigma_{f2}}{\rho_2}\right). \tag{12}$$

From Eq. (12) it can be seen that the material index is $p = \sigma_f/\rho$ which is consistent with previous results of Ashby [8] and others.

4.3. Vertical scaling: $u = 1$

When the width is constrained $u = 1$, and deriving v from Eq. (10) gives

$$v = \left(\frac{Z_2}{Z_1}\right)^{\frac{1}{2}}. \tag{13}$$

Eqs. (9) and (13) are combined with Eq. (7) to give the following ratio of the performance indices

$$\frac{p_2}{p_1} = \frac{\rho_1}{\rho_2}\left(\frac{Z_1}{Z_2}\right)^{\frac{1}{2}} = \left(\frac{\rho_1}{(\sigma_{f2})^{\frac{1}{2}}}\right)\left(\frac{(\sigma_{f1})^{\frac{1}{2}}}{\rho_2}\right). \tag{14}$$

From Eq. (14) it can be seen that the performance index is $p = \sigma_f^{1/2}/\rho$, which is consistant with previous results of Ashby [8] and others.

4.4. Arbitrary scaling, $u \neq 1$ and $v \neq 1$

For arbitrary scaling conditions $u \neq 1$ and $v \neq 1$. We seek a solution such that

$$\frac{p_2}{p_1} = \frac{\rho_1}{\rho_2}\left(\frac{Z_1}{Z_2}\right)^q = \left(\frac{\rho_1}{(\sigma_{f2})^q}\right)\left(\frac{(\sigma_{f1})^q}{\rho_2}\right), \tag{15}$$

where $q = f(D)$ is yet to be determined.

For $u \neq 1$ and $v \neq 1$ we write

$$\begin{cases} u = \left(\frac{Z_2}{Z_1}\right)^{\alpha}, \\ v = \left(\frac{Z_2}{Z_1}\right)^{\beta}. \end{cases} \tag{16}$$

Deriving the exponents α and β from Eq. (16) and using Eq. (10) gives

$$\begin{cases} \alpha = \lg_{\left(\frac{Z_2}{Z_1}\right)} u = \lg_{(uv^2)} u, \\ \beta = \lg_{\left(\frac{Z_2}{Z_1}\right)} v = \lg_{(uv^2)} v. \end{cases} \tag{17}$$

We now use Eq. (16) to rewrite Eq. (10) as

$$\frac{Z_2}{Z_1} = uv^2 = \left(\frac{Z_2}{Z_1}\right)^{\alpha+2\beta}. \tag{18}$$

The exponent q is given by

$$q = \alpha + 2\beta = \log_{(uv^2)} uv = \frac{\ln uv}{\ln uv^2}. \tag{19}$$

Eq. (19) represents the general solution to the exponent q of the performance index σ_f^q/ρ for the strength design of beams. As shown in Eq. (19) q is a function of the scaling parameters u and v and hence is a function of the direction of scaling.

Fig. 4 shows a plot of results of $q = f(u, v)$. These results are consistent with the previously determined results of $q = 2/3$, $q = 1$ and $q = 1/2$ for proportional scaling, constrained height and constrained width respectively. The figure shows that for two curves $uv^2 = 1$ and $uv = 1$, q is unbounded (i.e. $q = \pm\infty$) and zero respectively.

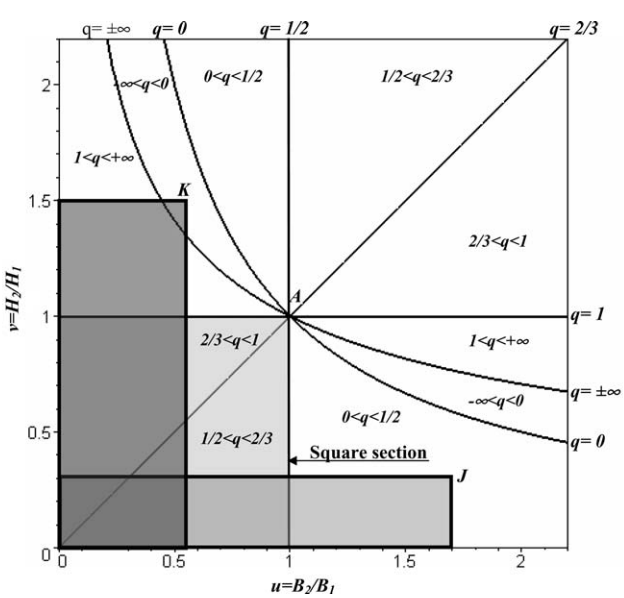

Fig. 4. Solution of q (Eq. (19)) for all directions of scaling.

The results give an indication about the relative importance of the strength and density of a particular material and for different directions of scaling. When a cross-section is scaled for low values of q (such as a width constraint where $q = 1/2$), then the density is relatively more important than the strength. In contrast, when the scaling is such that $q \geqslant 1$ (such as a horizontal or negatively sloped constraint as illustrated in Fig. 2(c)), strength is relatively more important compared to density. This shows that the direction of scaling has a very important effect on material selection.

Fig. 4 shows examples of arbitrarily scaled rectangular sections of different materials. The diagram shows how scaling of a cross-section changes the value of q and hence changes the material index. The diagram is split into regions for convenience so that a designer can immediately observe significant changes in the value of q. If a reference structure A, for instance, has a cross-section of unit dimensions, and is rescaled so that $u = 0.55$, $v = 1.5$, point A moves to point K and $-\infty < q < 0$. Alternatively, if point A moves to point J, then $0 < q < 1/2$. In both cases, q can be obtained from Eq. (19) and the material index can be calculated.

In the following section we show that materials with high σ_f, such as steel, perform relatively better for high values of q. In addition, we present useful q ranges where one material provides lower mass compared to others.

5. Material performance maps

The general solution to the performance index (Eqs. (1) and (19)) enables a comparison to be made between the performance of different materials for any direction of scaling. Examples of a full range of solutions for the performance index for three materials are shown in Fig. 5 with the material data shown in Table 1. The performance index has been plotted as a function of the scaling parameter q using values of σ_f and ρ. When the direction of scaling is known in a design task, q can be calculated from Eq. (19) and the relative performance of different materials can be immediately determined from Fig. 5.

The intersection points of two curves in Fig. 5 represent values of the scaling parameter q where different materials perform equally. Thus when $q > 1.28$, steel cross-sections have a better performance index than CFRP and GFRP cross-sections. When the scaling parameter $q < -0.05$, all GFRP rectangular cross-sections provide the best performance compared to steel and CFRP.

The parameter q is the scaling parameter. The relationship between u and v can be found by inverting Eq. (19) so that

$$v = u^{\frac{(1-q)}{(2q-1)}}. \tag{20}$$

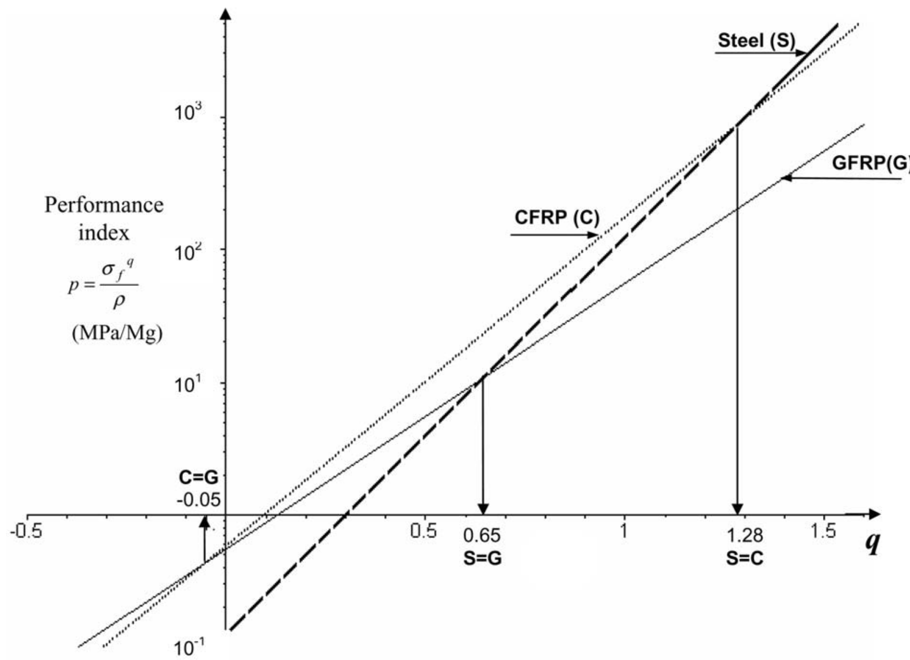

Fig. 5. Performance of three materials for a range of values of q. (CFRP = carbon fibre reinforced plastic; GFRP = glass fibre reinforced plastic.)

Table 1
Materials properties

Material	Bending failure strength σ_f (MPa)	Density ρ (Mg/m^3)
Steel	990	7.9
CFRP	300	1.7
GFRP	100	1.8

Curves of special q values for which two materials have the same performance index can be plotted and then limiting material regions mapped. These special values of q are plotted in Fig. 6 using Eq. (20). Fig. 6 shows regions where the performance of one material is relatively better compared to the others. For example, in strength design all the rectangular cross-sections manufactured

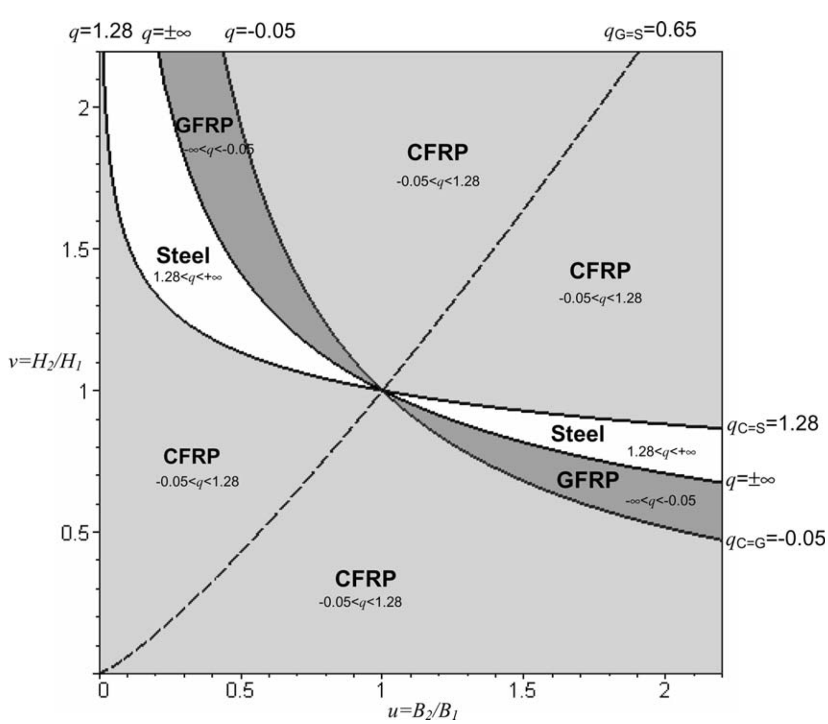

Fig. 6. Material performance map for Steel, CFRP, GFRP.

from steel provide the best performance index in the region where $q > 1.28$. In the case where $-0.05 < q < 1.28$, CFRP is always the best material, however GFRP is the second best choice only for $-0.05 < q < 0.65$, as shown in Fig. 5. Alternatively all cross-sections scaled so that they lie in the GFRP region always provide minimum mass compared to steel and CFRP. The design case given in the next section will show an application of the material performance maps.

6. Design case study

We now apply the performance index and the material performance maps to a practical case study shown in Fig. 7. A 1-metre cantilever beam has to be designed to support an end load of 50 KN without failing. Two candidate materials are considered: steel and CFRP. Both materials are constrained to have a rectangular cross-section. The size of the CFRP section is 10×10 cm in both cases. Two geometrical constraints on the design space are examined. One involves a height constraint with $q = 1$. The second involves a sloped constraint at an angle of 38° with $q = 1.44$. These conditions are shown in Fig. 7. Tables 2–4 present the data of the study.

Table 2
Design data for case study

	Failure strength σ_f (MPa)	Density ρ (Mg/m^3)	Length L (m)	End load P (KN)	Moment failure requirement M_f (KN m)
CFRP	300	1.7	1	50	50
Steel	990	7.9	1	50	50

6.1. Material performance map

The limiting regimes for the height constraint are illustrated in Fig. 8. In this case $q = 1$ and, for a given failure moment, a steel rectangular cross-section lies within the region where CFRP is better. Therefore CFRP provides lower mass for the horizontal height constraint.

The limiting regimes for the sloped constraint ($q = 1.44$) are illustrated in Fig. 9. The same CFRP section 10×10 cm is compared with a steel rectangular cross-section. The sloped constraint dictates the direction of scaling. Fig. 9 illustrates that this constraint

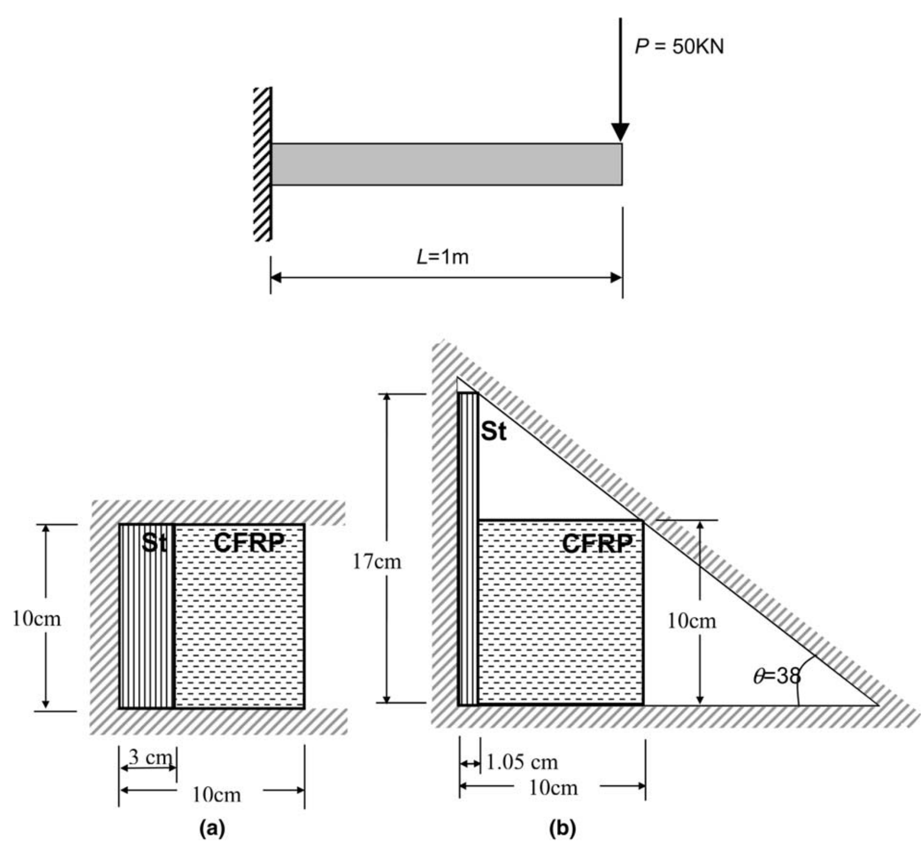

Fig. 7. The cantilever and its cross-sections in two different geometrically constrained conditions: (a) height constraint ($\theta = 0$, $q = 1$), (b) slope constraint ($\theta = 38$, $q = 1.44$).

Table 3
Results for the height constraint ($\theta = 0°$, $q = 1$)

	Width B (cm)	Height H (cm)	Width multiplier u	Height multiplier v	Power of performance index q	Performance index $p = \frac{\sigma_f^q}{\rho}$	Mass m (Mg)	Mass saving
CFRP	10	10	3	1	1.00	176.5	0.017	29%
Steel	3	10			1.00	125.3	0.0237	

Table 4
Results for the sloped constraint (slope $\theta = 38°$, $q = 1.44$)

	Width B (cm)	Height H (cm)	Width multiplier u	Height multiplier v	Power of performance index q	Performance index $p = \frac{\sigma_f^q}{\rho}$	Mass m (Mg)	Mass saving
CFRP	10	10	0.105	1.7	1.44	2225.9	0.017	
Steel	1.05	17			1.44	2687	0.014	17%

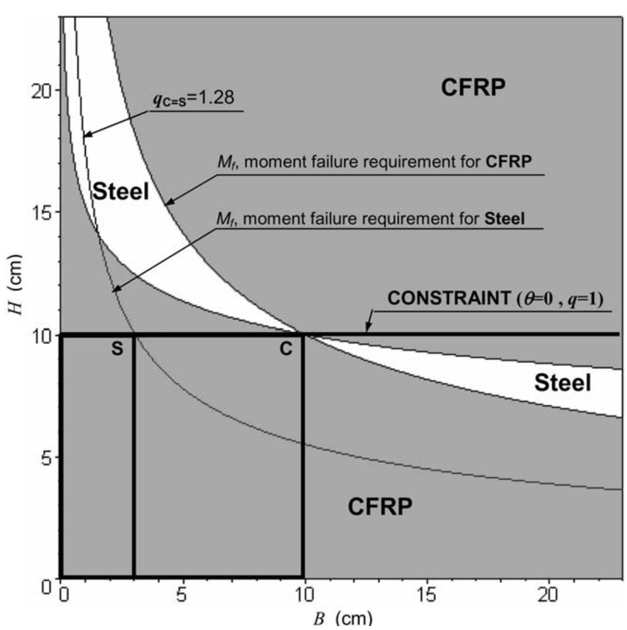

Fig. 8. Material performance map for a height constraint.

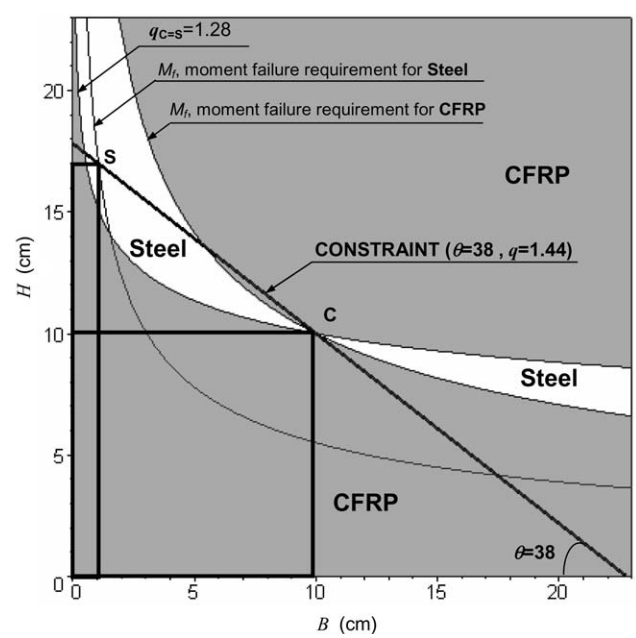

Fig. 9. Material performance map for a sloped constraint, $\theta = 38°$.

intersects the region where steel performs better than CFRP. This is in contrast to the first case, and the steel cross-section has lower mass than CFRP. The reversed ranking of steel and CFRP for the two constraints considered can be seen from Fig. 5 which shows that steel performs better than CFRP when $q > 1.28$.

The numerical results based on the general solution of the performance index are reported in Tables 3 and 4. For a horizontal constraint CFRP gives a mass saving of 29%. However, when there is a sloped constraint steel gives a mass saving of 17%. This case study demonstrates that the direction of scaling can have a very significant effect on the optimal choice of material. In particular, high-density materials like steel can perform surprisingly well in a confined space. This observation is consistent with the results of other related work [13,14].

7. Conclusion

Geometric constraints on the design space restrict the direction of scaling in material selection. Therefore, designers need a generalised material index that can cope with any arbitrary direction of scaling. The performance index for the bending strength design of beams has been previously only derived for proportional, horizontal and vertical scaling. In this paper we have adopted a methodology that can model the scaling of a cross-section when it is scaled in any arbitrary direction. The new methodology has enabled the general solution for the performance index for strength design to be determined.

Results of the general solution of the performance index have been plotted for steel, CFRP and GFRP. The results give an indication about the relative importance of the strength and density of a particular material and for different directions of scaling. When a cross-section is scaled for low values of q (such as a width constraint where $q = 1/2$), then the density is relatively more important than the strength. In contrast, when the scaling is such that $q \geqslant 1$ (such as a horizontal or negatively sloped constraint), strength is relatively more important compared to density. This shows that the direction of scaling has a very important effect on material selection. It also helps to explain why high-density materials like steel perform relatively well in tightly constrained spaces.

Acknowledgement

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